Constant k-curvature hypersurfaces in Riemannian manifolds

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ABSTRACT. In [8], Rugang Ye proved the existence of a family of constant mean curvature hypersurfaces in an $m+1$-dimensional Riemannian manifold (M^{m+1}, g), which concentrate at a point p_0 (which is required to be a nondegenerate critical point of the scalar curvature), moreover he proved that this family constitute a foliation of a neighborhood of p_0. In this paper we extend this result to the other curvatures (the r-th mean curvature for $1 \leq r \leq m$).

Key Words: Constant mean curvature, Foliations, Local Inversion.

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1 Introduction

Let S be an oriented embedded (or possibly immersed) hypersurface in a Riemannian manifold (M^{n+1}, g). The shape operator A_S is the symmetric endomorphism of the tangent bundle of S associated with the second fundamental form of S, b_S, by

$$b_S(X,Y) = g_S(A_S\, X, Y), \quad \forall\, X, Y \in TS; \qquad \text{here} \qquad g_S = g|_{TS}\,.$$

The eigenvalues κ_i of the shape operator A_S are the principal curvatures of the hypersurface S. The k-curvature of S is define to be the k-th symmetric function of the principal curvatures of S, i.e.

$$H_k(S) := \sum_{i_1 < \ldots < i_k} \kappa_{i_1} \ldots \kappa_{i_k}.$$

Hence, when $k = 1$, H_1 is equal to n times the mean curvature of S. When $M^{n+1} = \mathbb{R}^{n+1}$ is the Euclidean space, H_2 is equal to $\frac{n(n-1)}{2}$ times the scalar curvature of S and H_n is equal to the Gauss-Kronecker curvature of S. In this paper

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we are interested in the existence of hypersurfaces in M^{n+1} whose k-curvature is constant. Hypersurfaces with constant mean curvature, constant scalar curvature or constant Gauss-Kronecker curvature in Euclidean space or space forms constitute an important class of submanifolds. In Riemannian manifolds very few examples of constant k-curvature hypersurfaces are known, except when $k = 1$.

R. Ye [8], [9] has proved the existence of a local foliation by constant mean curvature hypersurfaces which concentrate at a point (which is required to be a nondegenerate critical point of the scalar curvature function). We extend the result and methods of [8] to handle the case $k = 2, \dots, n$. No extra curvature hypotheses are required. In particular, we prove the existence of foliations of a neighborhood of any nondegenerate critical point of the scalar curvature of (M^{n+1}, g) by constant Gauss-Kronecker or constant scalar curvature hypersurfaces. As in [8] the idea is to perturb $\bar{S}_\rho(p)$, a geodesic sphere with small radius $\rho > 0$ centered at a point p. A simple computation will show that $\bar{S}_\rho(p)$ is close to being a constant k-curvature hypersurface as ρ tends to 0 and in fact

$$\sigma_k(\bar{S}_\rho(p)) = C_n^k \, \rho^{-k} + \mathcal{O}(\rho^{-k+2}),$$

In this paper, we show that it is possible to perturb $\bar{S}_\rho(p)$ for every small radius, to a constant k-curvature hypersurface equal to $C_n^k \, \rho^{-k}$ for any $1 \leq k \leq n-1$, provided p is close to a nondegenerate critical point of the scalar curvature of M. The analysis here is inspired from the one performed in [8]. In fact, independently of the value of k, the linearized k-curvature operator about the unit Euclidean sphere is always a multiple of $\Delta_{S^n} + n$, the linearized mean curvature operator about the unit Euclidean sphere. This implies that, as in [8], to perform the perturbation of a small geodesic sphere, one has to overcome the problem of the existence of $(n+1)$-dimensional kernel of $\Delta_{S^n} + n$, kernel which is related to the invariance of k-curvature with respect to the action of isometries (in the case of the unit sphere, this kernel is only generated by translations). This is where, as in [8] we use the fact that we are close to a nondegenerate critical point of the scalar curvature of the ambient manifold.

We notice that the analysis performed in [8] is specific to treat the case of mean curvature, namely $k = 1$ and, unfortunately, can't be used to treat the general case $k = 2, \dots, n$. The main technical result of this paper is a precise expansion of geometric operators (first and second fundamental forms) for perturbed geodesic sphere (see Proposition 2.1, Proposition 3.1 and Proposition 3.2). We believe that these expansions are of independent interest and can be used in many other construction [4]. Our main result is :

Theorem 1.1. *Suppose that p_0 is a nondegenerate critical point of the scalar curvature $\mathcal{R}$ of M. Then there exists $\rho_0 > 0$, such that for all $\rho \in (0, \rho_0)$, the geodesic sphere $\bar{S}_\rho(p_0)$ may be perturbed to a constant k-curvature hypersurface S_ρ with $H_k = C_n^k \, \rho^{-k}$. Moreover these k-curvature hypersurfaces constitute a local foliation of a neighborhood of p_0.*

The existence of the hypersurfaces is not so difficult and can be obtained rather easily. The fact that they constitute a local foliation requires more work. The leaves S_ρ are small perturbation of geodesic spheres in the sense that S_ρ is a normal graph over $\bar{S}_\rho(p_0)$ for some function $\bar{w}_\rho$ which is bounded by a constant times ρ^2.

The hypersurface S_ρ is a small perturbation of $\bar{S}_\rho(p_0)$ in the sense that it is the normal graph of some function (with L^∞ norm bounded by a constant times ρ^2) over a geodesic sphere obtained centered at a point at distance bounded by a constant times ρ^2 of p_0.

Existence of families of constant mean curvature hypersurfaces concentrating along positive dimensional limit sets is obtained by R. Mazzeo and F. Pacard in [3] and then in collaboration with the author in [2] in a more general setting.

The paper is organized in the following way: In section 2 we expand the coefficients of the metric in normal geodesic coordinates. Section 3 will be devoted to the expansion of the first fondamental form, second fondamental form and the Shape operator of the perturbed geodesic spheres. Using these, we derive in section 4, the expansion of the k-curvature of the perturbed spheres. Section 5 is devoted to the proof of the main result of this paper, theorem 1.1.

2 Expansion of the metric in geodesic normal coordinates

In this Section we introduce geodesic normal coordinates in a neighborhood of a point $p \in M$. We choose an orthonormal basis E_i, $i = 1, \ldots, n+1$, of T_pM.

Consider, in a neighborhood of p in M, normal geodesic coordinates

$$F(x) := \exp_p^M(x_i\, E_i), \qquad x := (x_1, \ldots, x_{n+1}),$$

where $\exp^M$ is the exponential map on M and summation over repeated indices is understood. This yields the coordinate vector fields $X_i := F_*(\partial_{x_i})$. As usual, the Fermi coordinates above are defined so that the metric coefficients

$$g_{ij} = g(X_i, X_j)$$

equal δ_{ij} at p. We now compute higher terms in the Taylor expansions of the functions g_{ij}. The metric coefficients at $q := F(x)$ are given in terms of geometric data at $p := F(0)$ and $|x| := (x_1^2 + \ldots + x_{n+1}^2)^{1/2}$.

Notation The symbol $\mathcal{O}(|x|^r)$ indicates an analytic function such that it and its partial derivatives of any order, with respect to the vector fields $x^j\, X_i$, are bounded by a constant times $|x|^r$ in some fixed neighborhood of 0.

We now give the well known expansion for the metric in normal coordinates [1], [7], [11], but we will briefly recall the proof in the Appendix for completeness.

Proposition 2.1. *At the point $q = F(x)$, the following expansions hold*

$$\begin{aligned} g_{ij} &= \delta_{ij} + \frac{1}{3}\, g(R(E_k, E_i)\, E_\ell, E_j)\, x_k\, x_\ell \\ &+ \frac{1}{6}\, g(\nabla_{E_k} R(E_\ell, E_i) E_m, E_j)\, x_k\, x_\ell\, x_m + \mathcal{O}(|x|^4) \end{aligned} \tag{2.1}$$

where all curvature terms are evaluated at p. Convention ever repeated indices is understood.

3 Geometry of spheres

In this Section, we derive expansions as ρ tends to 0 for the metric, second fundamental form and mean curvature of the sphere $\bar{S}_\rho(p)$ and their perturbations.

Fix $\rho > 0$. We use a local parametrization $z \to \Theta(z)$ of $S^n \subset T_pM$. Now define the map

$$G(z) := F\left(\rho\,(1 - w(z))\,\Theta(z)\right),$$

and denote its image by $S_\rho(p, w)$, so in particular $S_\rho(p, 0) = \bar{S}_\rho(p)$. Because of the definition of these hypersurfaces using the exponential map, various vector fields we shall use may be regarded either as fields along $S_\rho(p, w)$ or as vectors of T_pM. To help allay this confusion, we write

$$\Theta := \Theta^j\, E_j \qquad\qquad \Theta_i := \partial_{z^i}\Theta^j\, E_j.$$

These are all vectors in the tangent space T_pM. On the other hand, the vectors

$$\Upsilon := \Theta^j\, X_j \qquad\qquad \Upsilon_i := \partial_{z^i}\Theta^j\, X_j$$

lie in the tangent space T_qM, where $q = F(z)$. For brevity, we also write

$$w_j := \partial_{z^j} w, \qquad w_{ij} := \partial_{z^i}\, \partial_{z^j} w.$$

In terms of all this notation, the tangent space to $S_\rho(w)$ at any point is spanned by the vectors

$$Z_j = G_*(\partial_{z^j}) = \rho\,((1 - w)\,\Upsilon_j - w_j\,\Upsilon), \qquad j = 1, \ldots, n. \tag{3.2}$$

Notation for error terms

The formulas for the various geometric quantities of $S_\rho(p, w)$ are potentially very complicated, and so it is important to condense notation as much as possible. Fortunately, we do not need to know the full structure of all of these quantities. Because it is so fundamental, we have isolated the notational conventions we shall use in this separate subsection.

Any expression of the form $L^j(w)$ denotes a linear combination of the functions w together with its derivatives with respect to the vector fields Θ_i up to order j.

The coefficients are assumed to be smooth functions on S^n which are bounded by a constant independent of $\rho \in (0,1)$ and $p \in M$, in $\mathcal{C}^\infty$ topology.

Similarly, any expression of the form $Q^j(w)$ denotes a nonlinear operator in the functions w together with its derivatives with respect to the vector fields Θ_i up to order j. Again, the coefficients of the Taylor expansion of the corresponding differential operator are smooth functions on S^n which are bounded by a constant independent of $\rho \in (0,1)$ and $p \in M$ in the $\mathcal{C}^\infty$ topology. In addition Q^j vanishes quadratically at $w = 0$.

Finally, any term of the form $L^j \ltimes Q^k$ will denote any finite sum of the product of a linear operators L^j with nonlinear operators Q^k.

We also agree that any term denoted $\mathcal{O}(\rho^d)$ is a smooth function on S^n which is bounded by a constant (independent of p) times ρ^d in the $\mathcal{C}^\infty$ topology.

The first fundamental form The next step is the computation of the coefficients of the first fundamental form of $S_\rho(p,w)$. We set $q := G(z)$ and $p := G(0)$. We obtain directly from (2.2) that

$$
\begin{aligned}
g(X_i, X_j) &= \delta_{ij} + \tfrac{1}{3}\, g(R(\Theta, E_i)\,\Theta, E_j)\,\rho^2\,(1-w)^2 \\
&+ \tfrac{1}{6}\, g(\nabla_\Theta R(\Theta, E_i)\,\Theta, E_j)\,\rho^3\,(1-w)^3 \\
&+ \mathcal{O}(\rho^4) + \rho^4\, L^0(w) + \rho^4\, Q^0(w).
\end{aligned}
\tag{3.3}
$$

where all the curvature terms are evaluated at p. Observe that we have

$$
g(\Upsilon, \Upsilon) \equiv 1 \qquad g(\Upsilon, \Upsilon_j) \equiv 0
$$

Using these expansions it is easy to obtain the expansion of the first fundamental form of $S_\rho(p,w)$.

Proposition 3.1. *We have*

$$
\begin{aligned}
&\rho^{-2}\,(1-w)^{-2}\, g(Z_i, Z_j) = g(\Theta_i, \Theta_j) + \tfrac{1}{3}\, g(R(\Theta, \Theta_i)\,\Theta, \Theta_j)\,\rho^2\,(1-w)^2 \\
&+ \tfrac{1}{6}\, g(\nabla_\Theta R(\Theta, \Theta_i)\,\Theta, \Theta_j)\,\rho^3\,(1-w)^3 + (1-w)^{-2}\, w_i\, w_j + \mathcal{O}(\rho^4) \\
&+ \rho^4 L^0(w) + \rho^4\, Q^0(w).
\end{aligned}
\tag{3.4}
$$

where all curvature terms are evaluated at p.

The normal vector field Our next task is to understand the dependence on w of the unit normal N to $S_\rho(w)$. Define the vector field

$$
\tilde{N} := -\,\Upsilon + A^j\, Z_j,
$$

and choose the coefficients A^j so that $\tilde{N}$ is orthogonal to all of the Z_i. This leads to a linear system for A^j.

$$
\sum_j A^j\, g(Z_j, Z_i) = -\rho\, w_i
$$

Observe that

$$g(\tilde{N}, \tilde{N}) = 1 + \rho \sum_j A_j \, w_j$$

The unit normal vector field N about $S_\rho(p, w)$ is defined to be

$$N := \frac{\tilde{N}}{g(\tilde{N}, \tilde{N})^{1/2}} \tag{3.5}$$

The second fundamental form We now compute the second fundamental form. To simplify the computations below, we henceforth assume that, at the point $\Theta(z) \in S^n$,

$$g(\Theta_i, \Theta_j) = \delta_{ij} \qquad \text{and} \qquad \overline{\nabla}_{\Theta_i} \Theta_j = 0, \quad i, j = 1, \ldots, n \tag{3.6}$$

(where $\overline{\nabla}$ is the connection on TS^{n-1}).

Proposition 3.2. *The following expansions hold*

$$\begin{aligned} -g(\nabla_{Z_i} N, Z_j) \quad &= \quad \rho\,(1-w)\,\delta_{ij} + \rho\, w_{ij} + \tfrac{2}{3}\, g(R(\Theta, \Theta_i)\,\Theta, \Theta_j)\,\rho^3\,(1-w)^3 \\ &+ \quad \tfrac{5}{12}\, g(\nabla_\Theta R(\Theta, \Theta_i)\,\Theta, \Theta_j)\,\rho^4\,(1-w)^4 \\ &- \quad \tfrac{1}{3}\,\left(g(R(\nabla w,\, \Theta_i)\,\Theta, \Theta_j) + g(R(\Theta,\, \Theta_i)\,\nabla w, \Theta_j)\right)\,\rho^3 \\ &+ \quad \mathcal{O}(\rho^5) + \rho^4\, L^1(w) + \rho\, Q^1(w) + \rho\, L^2(w) \ltimes Q^1(w) \end{aligned} \tag{3.7}$$

where as usual, all curvature terms are computed at the point p.

Proof : We will first obtain the expansion of $g(\nabla_{Z_i} \tilde{N}, Z_j)$. To this aim, we compute

$$\begin{aligned} - &\quad g(\nabla_{Z_i} \tilde{N}, Z_j) \\ = &\quad g(\nabla_{Z_i} \Upsilon, Z_j) - \textstyle\sum_k g(\nabla_{Z_i} (A^k\, Z_k), Z_j) \\ = &\quad \tfrac{1}{1-w} g(\nabla_{Z_i} ((1-w) \Upsilon), Z_j) + \tfrac{1}{1-w}\, w_i\, g(\Upsilon, Z_j) - \textstyle\sum_k g(\nabla_{Z_i} (A^k\, Z_k), Z_j) \\ = &\quad \tfrac{1}{1-w} g(\nabla_{Z_i} ((1-w)\, \Upsilon), Z_j) - \tfrac{\rho}{1-w}\, w_i\, w_j - \textstyle\sum_k g(\nabla_{Z_i} (A^k\, Z_k), Z_j) \end{aligned}$$

Now, recall that

$$\sum_k A^k\, g(Z_k, Z_j) = -\rho\, w_j$$

Hence

$$\sum_k g(\nabla_{Z_i} (A^k\, Z_k), Z_j) = -\rho\, w_{ij} - \sum_k A^k\, g(Z_k, \nabla_{Z_i} Z_j)$$

Using the fact that

$$2\, g(Z_k, \nabla_{Z_i} Z_j) = Z_i\, g(Z_k, Z_j) + Z_j\, g(Z_k, Z_i) - Z_k\, g(Z_i, Z_j)$$

we conclude that

$$\sum_k g(\nabla_{Z_i}(A^k Z_k), Z_j) \quad = \quad -\rho w_{ij} - \frac{1}{2}\sum_k A^k \Big(Z_i g(Z_k, Z_j) + Z_j g(Z_k, Z_i) \\ - \quad Z_k g(Z_i, Z_j) \Big)$$

To analyze the term $\nabla_{Z_i}((1-w)\,\Upsilon)$, let us revert for the moment and regard w as functions of the coordinates z and also consider ρ as a variable instead of just a parameter. Thus we consider

$$\tilde{F}(\rho, z) = F\big(\rho(1 - w(z))\Theta(z)\big).$$

The coordinate vector fields Z_j are still equal to $\tilde{F}_*(\partial_{z_j})$, but now we also have $Z_0 := (1-w)\,\Upsilon = \tilde{F}_*(\partial_\rho)$, which is the identity we wish to use below. Now, we write

$$\begin{aligned} & g(\nabla_{Z_i}((1-w)\,\Upsilon), Z_j) + g(\nabla_{Z_j}((1-w)\,\Upsilon), Z_i) \\ = \quad & g(\nabla_{Z_i} Z_0, Z_j) + g(\nabla_{Z_j} Z_0, Z_i) \\ = \quad & Z_0\, g(Z_i, Z_j) \end{aligned}$$

Collecting the above we have obtained to formula

$$\begin{aligned} -g(\nabla_{Z_i}\tilde{N}, Z_j) \quad = \quad & \tfrac{1}{2(1-w)}\, Z_0\, g(Z_i, Z_j) - \tfrac{1}{1-w}\, \rho\, w_i\, w_j + \rho\, w_{ij} \\ + \quad & \tfrac{1}{2}\sum_k A^k\, \left(Z_i\, g(Z_k, Z_j) + Z_j\, g(Z_k, Z_i) - Z_k\, g(Z_i, Z_j)\right) \end{aligned}$$

We will now expand the first and last term in this expression.

If the coordinates y are chosen so that $g(\Theta_i, \Theta_j) = \delta_{ij}$ at the point where we will compute the shape form, we have, using the result of Proposition 3.1,

$$\begin{aligned} \tfrac{1}{2(1-w)}\, Z_0\, g(Z_i, Z_j) \quad = \quad & \rho\,(1-w)\,\delta_{ij} + \tfrac{2}{3}\, g(R(\Theta, \Theta_i)\,\Theta, \Theta_j)\,\rho^3\,(1-w)^3 \\ + \quad & \tfrac{5}{12}\, g(\nabla_\Theta R(\Theta, \Theta_i)\,\Theta, \Theta_j)\,\rho^4\,(1-w)^4 + \tfrac{1}{1-w}\,\rho\, w_i\, w_j \\ + \quad & \mathcal{O}(\rho^5) + \rho^5\, L^0(w) + \rho^5\, Q^0(w). \end{aligned}$$

Using the same Proposition together with the fact that the coordinates y are chosen so that $\bar{\nabla}_{\Theta_i}\Theta_j = 0$ at the point where we will compute the shape form, we also have

$$\begin{aligned} & Z_i\, g(Z_k, Z_j) + Z_j\, g(Z_k, Z_i) - Z_k\, g(Z_i, Z_j) = \\ & \tfrac{2}{3}\, \left(g(R(\Theta_k, \Theta_i)\,\Theta_j, \Theta) + g(R(\Theta_k, \Theta_i)\,\Theta_j, \Theta)\right)\, \rho^4 \\ & + \mathcal{O}(\rho^5) + \rho^2\, L^1(w) + \rho^2\, Q^1(w) + \rho^2\, L^2(w) \ltimes L^1(w) \end{aligned}$$

If the coordinates y are chosen so that $g(\Theta_i, \Theta_j) = \delta_{ij}$ at the point where we will compute the shape form, we have the expansion

$$A^k = -\frac{w_k}{\rho(1-w)^2} + \rho\, L^1(w) + \rho\, Q^1(w)$$

collecting the above estimates, we conclude that

$$\begin{aligned}
-g(\nabla_{Z_i}\tilde{N}, Z_j) &= \rho\,(1-w)\,\delta_{ij} + \rho\, w_{ij} + \tfrac{2}{3}\, g(R(\Theta,\Theta_i)\,\Theta,\Theta_j)\,\rho^3\,(1-w)^3 \\
&+ \tfrac{5}{12}\, g(\nabla_\Theta R(\Theta,\Theta_i)\,\Theta,\Theta_j)\,\rho^4\,(1-w)^4 \\
&- \tfrac{1}{3}\,\left(g(R(\Theta_k,\,\Theta_i)\,\Theta,\Theta_j) + g(R(\Theta,\,\Theta_i)\,\Theta_k,\Theta_j)\right)\,\rho^3\, w_k \\
&+ \mathcal{O}(\rho^5) + \rho^4\, L^1(w) + \rho\, Q^1(w) + \rho\, L^2(w) \ltimes Q^1(w)
\end{aligned}$$

It remains to observe that

$$g(\tilde{N},\tilde{N})^{-1/2} = 1 + Q^1(w)$$

This finishes the proof of the estimate. □

The shape operator of perturbed surfaces Collecting the estimates of the last subsection we obtain the expansion of the shape operator of the hypersurface $S_\rho(p,w)$. In the coordinate system defined in the previous sections, we get

Proposition 3.3. *Under the previous hypothesis, the shape operator of the hypersurface $S_\rho(p,w)$ is given by*

$$\begin{aligned}
\rho\, A_{ij}(w) &= (1+w)\,\delta_{ij} + w_{ij} + \tfrac{1}{3}\, g(R(\Theta,\,\Theta_i)\,\Theta,\Theta_j)\,\rho^2 \\
&- \tfrac{1}{3}\,\big[\, g(R(\Theta,\,\Theta_i)\,\Theta,\Theta_j)\, w + g(R(\Theta,\Theta_i),\Theta,\Theta_k)\, w_{kj} \\
&+ \left(g(R(\Theta_k,\,\Theta_i)\,\Theta,\Theta_j) + g(R(\Theta,\,\Theta_i)\,\Theta_k,\Theta_j)\right)\, w_k\big]\,\rho^2 \\
&+ \tfrac{1}{4}\, g(\nabla_\Theta R(\Theta,\,\Theta_i)\,\Theta,\Theta_j)\,\rho^3 + \mathcal{O}(\rho^4) + \rho^3 L^2(w) \\
&+ Q^1(w) + L^2(w) \ltimes L^0(w) + L^2(w) \ltimes Q^1(w).
\end{aligned}$$

where all curvature terms are computed at the point p.

4 The k-curvature of the perturbed sphere

Given any symmetric matrix A, and any $k = 0,\ldots,n$, we define

$$\sigma_k(A) := \sum_{i_1<\ldots<i_k} \lambda_{i_1}\ldots\lambda_{i_k}.$$

where $\lambda_1,\ldots,\lambda_n$ are the eigenvalues of A. The k-th Newton transform of A is defined by

$$T_k(A) := \sigma_k(A)\, I - \sigma_{k-1}(A)\, A + \cdots + (-1)^k\, A^k.$$

with $T_n(A) = 0$. Now suppose that $A = A(t)$ depends smoothly on a parameter t, it is proved in [5] that

$$\frac{d}{dt}\sigma_k(A) = \mathrm{Tr}\left(T_{k-1}(A)\,\frac{d}{dt}A\right) \tag{4.8}$$

From this computation, it follows at once that, given any $n \times n$ symmetric matrix H,

$$\sigma_k(I + H) = C_n^k + C_{n-1}^{k-1}\,\mathrm{Tr}(H) + \mathcal{O}(|H|^2)$$

Using this together with the previous expansion of the shape operator, it is not hard to check that the k-curvature of the hypersurface $S_\rho(p, w)$ can be expanded as

$$\begin{aligned}
& \rho^k\, H_k(S_\rho(p,w)) \\
= \quad & C_n^k + C_{n-1}^{k-1}\,\big[(\Delta_{S^n} + n)\, w - \tfrac{1}{3}\,\mathrm{Ric}(\Theta,\Theta)\,\rho^2 - \tfrac{1}{4}\,\nabla_\Theta \mathrm{Ric}(\Theta,\Theta)\,\rho^3 \\
+ \quad & \tfrac{1}{3}\,\big(\mathrm{Ric}(\Theta,\Theta) + 2\,\mathrm{Ric}(\nabla\cdot,\Theta) - g(R(\Theta,\nabla\cdot)\Theta,\nabla\cdot)\big)\; w\,\rho^2 \\
+ \quad & \mathcal{O}(\rho^4) + \rho^3\, L^2(w) + Q^1(w) + L^2(w) \ltimes L^0(w) + L^2(w) \ltimes Q^1(w)\big]
\end{aligned}$$

where as usual, all curvature terms are computed at p. Here we have defined

$$\mathrm{Ric}(\nabla\cdot,\Theta) := \mathrm{Ric}(e_i,\Theta)\, e_i$$

and

$$g(R(\Theta,\nabla\cdot),\Theta,\nabla\cdot) := g(R(\Theta,e_i),\Theta,e_j)\, e_i\, e_j$$

if $e_1, \ldots, e_n$ is an orthonormal frame field of $T_{\bar q}S^n$ satisfying $\bar\nabla_{e_i}\, e_j = 0$ at the point $\bar q \in S^n$ where these expressions are computed. It will be convenient to set

$$\mathcal{L} := \frac{1}{3}\;\big(\mathrm{Ric}(\Theta,\Theta) + 2\,\mathrm{Ric}(\nabla,\Theta) - g(R(\Theta,\nabla),\Theta,\nabla)\big)$$

Now observe that a similar expansion is valid in Euclidean space and in this case the expansion of $\rho^{-k}\, H_k(p,w)$ does not depend on ρ (nor on p). This means that the nonlinear operator

$$Q^2 := Q^1 + L^2 \ltimes L^0 + L^2 \ltimes Q^1$$

can be decomposed into its value in Euclidean space and a similar operator all of whose coefficients are bounded by ρ. This fact can also be recovered by going through all the above expansions. Therefore, we can write

$$Q^2 = Q_e^2 + \rho\, Q_r^2$$

where Q_e^2 is the corresponding nonlinear operator when the metric is Euclidean and hence it does not depend on ρ; while $\rho\, Q_r^2$ denotes the discrepancy induced by the curvature of the metric g on M. Both Q_e^2 and Q_r^2 satisfy the usual properties.

5 Existence of foliations by constant k-curvature hypersurfaces

Assume that we are given $p_0 \in M$, a nondegenerate critical point of the scalar curvature $\mathcal{R}$ on M. We would like to find a small function $w \in \mathcal{C}^{2,\alpha}(S^n)$ and a point p close to p_0 such that

$$H_k(S_\rho(p,w)) = C_n^k\, \rho^{-k}$$

In view of the previous expansion, this amount to solve the nonlinear equation

$$
\begin{aligned}
(\Delta_{S^n} + n)\, w = \frac{1}{3}\,\mathrm{Ric}(\Theta,\Theta)\,\rho^2 + \frac{1}{4}\,\nabla_\Theta \mathrm{Ric}(\Theta,\Theta)\,\rho^3 - \mathcal{O}(\rho^4) \\
- \rho^2\,\mathcal{L}\,w - \rho^3\,L^2(w) - Q^2(w)
\end{aligned}
\tag{5.9}
$$

We denote by Π and $\Pi^\perp$ the L^2-orthogonal projections of $L^2(S^n)$ onto $\mathrm{Ker}(\Delta_{S^n} + n)$ and $\mathrm{Ker}(\Delta_{S^n} + n)^\perp$, respectively. Recall that the kernel of $\Delta_{S^n} + n$ is spanned by φ_i, for $i = 1, \ldots, n+1$, the restriction to the unit sphere of x_i, the coordinates functions in $\mathbb{R}^{n+1}$.

First fixed point argument From now on, we assume that the function $w \in \mathcal{C}^{2,\alpha}(S^n)$ is L^2-orthogonal to $\mathrm{Ker}(\Delta_{S^n} + n)$ and we project the equation (5.9) over $\mathrm{Ker}(\Delta_{S^n} + n)^\perp$. We obtain

$$
\begin{aligned}
(\Delta_{S^n} + n)\, w = \quad & \Pi^\perp\,\left[\tfrac{1}{3}\,\mathrm{Ric}(\Theta,\Theta)\,\rho^2 + \tfrac{1}{4}\,\nabla_\Theta \mathrm{Ric}(\Theta,\Theta)\,\rho^3 - \mathcal{O}(\rho^4)\right. \\
& \left. -\rho^2\,\mathcal{L}\,w - \rho^3\,L^2(w) - Q^2(w)\right]
\end{aligned}
$$

We define $w_0 \in \mathrm{Ker}(\Delta_{S^n} + n)^\perp$ to be the unique solution of

$$
(\Delta_{S^n} + n)\, w_0 = \frac{1}{3}\,\mathrm{Ric}(\Theta,\Theta) \tag{5.10}
$$

since $\Pi^\perp\,(\mathrm{Ric}(\Theta,\Theta)) = \mathrm{Ric}(\Theta,\Theta)$. Similarly, we define $w_1 \in \mathrm{Ker}(\Delta_{S^n} + n)^\perp$ to be the unique solution of

$$
(\Delta_{S^n} + n)\, w_1 = \frac{1}{4}\,\Pi^\perp\,[\nabla_\Theta \mathrm{Ric}(\Theta,\Theta)]
$$

It is easy to rephrase the solvability of the nonlinear equation (5.9) as a fixed point problem since the operator $\Delta_{S^n} + n$ is invertible from the space of $\mathcal{C}^{2,\alpha}(S^n)$ functions which are L^2-orthogonal to $\mathrm{Ker}(\Delta_{S^n} + n)$ into the space of $\mathcal{C}^{0,\alpha}(S^n)$ functions which are L^2-orthogonal to $\mathrm{Ker}(\Delta_{S^n} + n)$. We write $w := \rho^2\, w_0 + \rho^3\, w_1 + \rho^4\, v$, so that it remains to solve an equation which can be written for short as

$$
(\Delta_{S^n} + n)\, v = -\mathcal{O}(1) - \rho^{-2}\,\mathcal{L}w - \rho^{-1}\,L^2(w) - \rho^{-4}\,Q^2(w)
$$

Applying a standard fixed point theorem for contraction mappings, it is easy to check that there exists a constant $\kappa > 0$, which is independent of the choice of the point $p \in M$, such that there exists a unique fixed point in ball of radius κ in $\mathcal{C}^{2,\alpha}(S^n)$, provided ρ is chosen small enough, say $\rho \in (0, \rho_0)$. We denote by v_p this solution and define

$$
w_p := \rho^2\, w_0 + \rho^3\, w_1 + \rho^4\, v_p.
$$

It is easy to check that, reducing the value of ρ_0 if this is necessary,

$$
\|w_p - w_{p'}\|_{\mathcal{C}^{2,\alpha}(S^n)} \leq c\,\rho^2\,\mathrm{dist}(p, p'), \tag{5.11}
$$

for some constant c which does not depend on $\rho \in (0, \rho_0)$ nor on p or p'. In addition, the mapping

$$(\rho, p) \in (0, \rho_0) \times M \longrightarrow w_p \in \mathcal{C}^{2,\alpha}(S^n)$$

is smooth and

$$\|D_p w_p\|_{\mathcal{C}^{2,\alpha}(S^n)} + \rho \, \|\partial_\rho w_p\|_{\mathcal{C}^{2,\alpha}(S^n)} \leq c \, \rho^2$$

for some constant c which does not depend on $\rho \in (0, \rho_0)$ nor on p.

Second fixed point argument It now remains to project the equation (5.9) where w has been replaced by w_p, over $\mathrm{Ker}(\Delta_{S^n} + n)$. To this aim, we recall the nice and key observation from [8].

The problem is to compute the L^2-projection of the term $g(\nabla_\Theta R(\Theta, \Theta_i)\Theta, \Theta_j)$ over the kernel of the operator $\Delta_{S^n} + n$. This amounts to compute, for any $m = 1, \ldots, n+1$, the quantity

$$B_m := \sum_{i,j,k,\ell} g(\nabla_{E_j} R(E_i, E_k) E_i, E_\ell) \int_{S^n} x_j \, x_k \, x_\ell \, x_m$$

Now to evaluate this quantity, simply use the fact that the integral vanishes unless all indices are all equal or constitute two pairs of equal indices. Using this, together with the symmetries of the curvature tensor which imply that $R(E, E) = 0$, we obtain

$$\begin{aligned} B_m &= g(\nabla_{E_m} R(E_i, E_m) E_i, E_m) \left(\textstyle\int_{S^n} x_1^4 - 3 \int_{S^n} x_1^2 \, x_2^2 \right) \\ &+ g(\nabla_{E_m} R(E_i, E_j) E_i + 2 \nabla_{E_j} R(E_i, E_m) E_i, E_j) \textstyle\int_{S^n} x_1^2 \, x_2^2 \end{aligned}$$

Now, use second Bianchi identity

$$g(\nabla_{E_m} R(E_i, E_j) E_i, E_j) = 2 \, g(\nabla_{E_j} R(E_i, E_m) E_i, E_j)$$

together with the fact that

$$\int_{S^n} x_1^4 = 3 \int_{S^n} x_1^2 \, x_2^2 = \frac{3}{(n+3)} \int_{S^n} x_1^2$$

To conclude that

$$\Pi \left(g(\nabla_\Theta R(\Theta, \, \Theta_i) \, \Theta, \Theta_j) \right) = -\frac{1}{n+3} \, g(\nabla \mathcal{R}, x_i \, E_i)$$

where $\mathcal{R}$ denotes the scalar curvature function, computed at p.

Therefore, the projection of the equation (5.9) over Ker $(\Delta_{S^n} + n)$ yields

$$g(\nabla \mathcal{R}, x_i \, E_i) = V_p$$

where we have defined

$$V_p := 4 \, (n+3) \, \Pi \, \left[\rho^{-3} \, \mathcal{O}(\rho^4) + \rho^{-1} \, \mathcal{L} \, w_p + L^2(w_p) + \rho^{-3} \, Q^2(w_p) \right]$$

Now, using the fact that p_0 is a nondegenerate critical point of the scalar curvature, we conclude easily (applying for example a topological degree argument) that there exists p close to p_0 satisfying (5.11) provided ρ is close enough to 0. This gives the existence of constant k-curvature leaves for all ρ small enough, unfortunately it turns out that the point p is at most at distance a constant times ρ from p_0 and this is not enough to show that the constant k-curvature leaves form a foliation of a neighborhood of p_0.

To improve this estimate, many observations are due. First, observe that we can decompose $\mathcal{O}(\rho^4)$ into the sum of two functions, one of which is homogeneous of degree 4 (in the coordinate functions x_i) and the other one which is bounded by a constant times ρ^5. The L^2-projection of the homogeneous function of degree 4 is equal to 0 since this homogeneous function is invariant under the change of coordinates Θ into $-\Theta$. Hence we conclude that

$$|\Pi\,(\mathcal{O}(\rho^4))| \leq c\,\rho^5$$

Similarly, observe that w_0 and hence $\mathcal{L}\,w_0$ are invariant under the change Θ into $-\Theta$ and hence the L^2 projection of $\mathcal{L}\,w_0$ over $\mathrm{Ker}(\Delta_{S^n}+n)$ again identically equal to 0. Therefore, we conclude that

$$|\Pi\,(\mathcal{L}\,w_p)| \leq c\,\rho^3$$

Finally, we use the observation at the end of §4. Since the nonlinear operator Q_e^2 preserves functions which are invariant under the action of $-I$, we conclude that $\Pi(Q_e^2(\rho^2\,w_0)) = 0$ and hence

$$|\Pi(Q^2(w_p))| \leq c\,\rho^5$$

These precise estimates imply that,

$$|V_p| \leq c\,\rho^2$$

for some constant which does not depend on p nor on ρ. With slightly more work, we get using similar arguments that

$$|\Pi\;(V_p - V_{p'})\,| \leq c\,\rho^2\,\mathrm{dist}(p,p') \tag{5.12}$$

Now, for all ρ small enough, we can find a solution of (5.9) using a fixed point argument for contraction mapping, in the geodesic ball of radius $2\,\rho^2$ centered at any nondegenerate critical point of $\mathcal{R}$. Moreover, the solution p_ρ depends smoothly on ρ and

$$|\partial_\rho p_\rho| \leq c\,\rho$$

This later fact, together with (5.12) shows that the solutions constitute a local foliation. This completes the proof of the main result. □

Having derived such precise estimates, we can compute the expansion of the n-dimensional volume of the leaves of the foliation as well as the $(n+1)$-dimensional volume enclosed by each leaf.

Proposition 5.1. *For all ρ small enough the following expansions hold for the n-dimensional volume of S_ρ*

$$Vol_n(S_\rho) = \rho^n \; Vol_n(S^n) \left(1 - \frac{1}{2(n+1)} \mathcal{R}\, \rho^2 + \mathcal{O}(\rho^4)\right)$$

and the $(n+1)$-dimensional volume of the set B_ρ enclosed by S_ρ and containing the point p_0

$$Vol_{n+1}(B_\rho) = \frac{1}{n+1} \, \rho^{n+1} \; Vol_n(S^n) \left(1 - \frac{n+2}{2n(n+3)} \mathcal{R}\, \rho^2 + \mathcal{O}(\rho^4)\right)$$

where the scalar curvature is computed at p_0, a nondegenerate critical point of $\mathcal{R}$.

Proof : Integrating (5.10) over S^n we find

$$n \int_{S^n} w_0 = \frac{1}{3} \int_{S^n} \mathrm{Ric}(\Theta, \Theta)$$

Now, plugging the expansion of w_p into the expression of the first fundamental form given in Proposition 3.1, we find the expansion of h the induced metric on S_ρ

$$\begin{aligned} \rho^{-2}\, h_{ij} &= (1 - 2\rho^2 w_0 - 2\rho^3 w_1)\delta_{ij} + \frac{1}{3}\, g(R(\Theta, \Theta_i)\,\Theta, \Theta_j)\,\rho^2 \\ &\quad + \frac{1}{6}\, g(\nabla_\Theta R(\Theta, \Theta_i)\,\Theta, \Theta_j)\,\rho^3 + \mathcal{O}(\rho^4) \end{aligned} \tag{5.13}$$

This implies that

$$\rho^{-n} \sqrt{|h|} = 1 - n\,\rho^2\, w_0 - n\,\rho^3\, w_1 - \frac{1}{6}\,\mathrm{Ric}(\Theta, \Theta)\,\rho^2 - \frac{1}{12}\,\nabla_\Theta \mathrm{Ric}(\Theta, \Theta)\,\rho^3 + \mathcal{O}(\rho^4)$$

The first estimate follows from integrating this expansion using the fact that the integral of w_1 and the integral $\nabla_\Theta \mathrm{Ric}\,(\Theta, \Theta)$ over S^n vanish together with the fact that

$$\int_{S^n} \mathrm{Ric}(\Theta, \Theta) = \frac{1}{n+1}\, \mathrm{Vol}_n(S^n)\, \mathcal{R}$$

Next, we consider polar geodesic normal coordinates (r, Θ) centered at p_ρ. In these coordinates the metric g expanded as

$$r^{-2}\, g_{ij} = \delta_{ij} + \frac{1}{3}\, g(R(\Theta, \Theta_i)\,\Theta, \Theta_j)\, r^2 + \frac{1}{6}\, g(\nabla_\Theta R(\Theta, \Theta_i)\,\Theta, \Theta_j)\, r^3 + \mathcal{O}(r^4). \tag{5.14}$$

then, the volume form can be expanded as

$$r^{-n}\, \sqrt{|g|} = 1 - \frac{1}{6}\,\mathrm{Ric}(\Theta, \Theta)\, r^2 - \frac{1}{12}\,\nabla_\Theta \mathrm{Ric}(\Theta, \Theta)\, r^3 + \mathcal{O}(r^4)$$

Integration over the set $r \le \rho\,(1-w_p)$ give

$$
\begin{aligned}
&\mathrm{Vol}_{n+1}(B_\rho) \\
&= \int\int_{r\le\rho\,(1-w_p)} r^n \left(1 - \frac{1}{6}\,\mathrm{Ric}(\Theta,\Theta)\,r^2 - \frac{1}{12}\,\nabla_\Theta \mathrm{Ric}(\Theta,\Theta)\,r^3 + \mathcal{O}(r^4)\right) \\
&= \frac{1}{n+1}\,\rho^{n+1}\int_{S^n}(\,1-(n+1)\rho^2\,w_0\,) + \mathcal{O}(\rho^{n+5}) \\
&- \frac{1}{6}\,\frac{1}{n+3}\,\rho^{n+3}\int_{S^n}(\,1-(n+3)\rho^2\,w_0\,)\,\mathrm{Ric}(\Theta,\Theta) \\
&= \frac{1}{n+1}\rho^{n+1}\mathrm{Vol}_n(S^n) - \rho^{n+3}\int_{S^n} w_0 - \frac{1}{6}\,\frac{\rho^{n+3}}{n+3}\int_{S^n}\mathrm{Ric}(\Theta,\Theta) + \mathcal{O}(\rho^{n+5}) \\
&= \frac{1}{n+1}\rho^{n+1}\mathrm{Vol}_n(S^n) - (\,\frac{1}{3n} + \frac{1}{6}\,\frac{1}{n+3})\rho^{n+3}\int_{S^n}\mathrm{Ric}(\Theta,\Theta) + \mathcal{O}(\rho^{n+5}) \\
&= \frac{1}{n+1}\,\rho^{n+1}\,\mathrm{Vol}_n(S^n)\,\left(1 - \frac{n+2}{2n(n+3)}\,\mathcal{R}\,\rho^2 + \mathcal{O}(\rho^4)\right)
\end{aligned}
$$

This gives the second estimate. This proves the desired result. □

6 Appendix : proof of Proposition 2.1

The aim of this Section is to prove Proposition 2.1. Observe first that the curve $s \to \exp_p^M(sE)$ is a geodesic. Therefore, if X is the unit tangent vector to the curve we have $\nabla_X X = 0$. Hence we also have $(\nabla_X)^m X = 0$ for all $m \ge 1$. In particular, we have, at p,

$$(\nabla_E)^m E = 0$$

for all $E \in T_pM$ and for all $m \ge 1$.

Observe that X_a are coordinate vector fields hence

$$\nabla_{X_a} X_b = \nabla_{X_b} X_a$$

Taking $E = E_a + \varepsilon\,E_b$ and looking for the coefficient of ε in $\nabla_E E = 0$, we get

$$\nabla_{E_a} E_b = 0$$

Looking at the coefficient of ε in $\nabla_E^2 E = 0$, we get

$$2\,\nabla_{E_a}^2 E_b + \nabla_{E_b}\nabla_{E_a} E_a = 0 \tag{6.15}$$

Finally, looking at the coefficient of ε in $\nabla_E^3 E = 0$, we get

$$2\,\nabla_{E_a}^3 E_b + (\nabla_{E_b}\nabla_{E_a} + \nabla_{E_a}\nabla_{E_b})\nabla_{E_a} E_a = 0 \tag{6.16}$$

Recall that, by definition

$$\nabla_X \nabla_Y := R(X,Y) + \nabla_Y \nabla_X + \nabla_{[X,Y]}$$

Hence, if X and Y are coordinate vector fields we simply have

$$\nabla_X \nabla_Y X := R(X,Y)X + \nabla_Y \nabla_X X \tag{6.17}$$

We also have

$$\begin{aligned} \nabla_Y \nabla_X \nabla_Y X \quad &:= \quad \nabla_Y R(X,Y)X + R(\nabla_Y X, Y)X + R(X, \nabla_Y Y)X \\ &+ \quad R(X,Y)\nabla_Y X + \nabla_Y^2 \nabla_X X + \nabla_Y \nabla_{[X,Y]} X \end{aligned} \tag{6.18}$$

Now use (6.15) and (6.17) to obtain

$$3\,\nabla^2_{E_a} E_b = R(E_a, E_b)\, E_a, \tag{6.19}$$

Similarly, use (6.16) and (6.18) to obtain

$$2\,\nabla^3_{E_a} E_b + R(E_b, E_a)\,\nabla_{E_a} E_a + 2\nabla_{E_a}\nabla_{E_b}\nabla_{E_a} E_a = 0 \tag{6.20}$$

Since $\nabla_{E_a} E_b = 0$, we get

$$2\,\nabla^3_{E_a} E_b + 2\nabla_{E_a}\nabla_{E_b}\nabla_{E_a} E_a = 0$$

Using this, we conclude that

$$\begin{aligned} 2\,\nabla^3_{E_a} E_b = -2\,\nabla_{E_a}\nabla_{E_b}\nabla_{E_a} E_a &= -2\,\nabla_{E_a}(R(E_b, E_a)\, E_a + \nabla_{E_a}\nabla_{E_a} E_b) \\ &= -2\,\nabla_{E_a}(R(E_b, E_a)\, E_a) - 2\,\nabla^3_{E_a} E_b \end{aligned}$$

Hence

$$2\,\nabla^3_{E_a} E_b = -\nabla_{E_a} R(E_b, E_a)\, E_a \tag{6.21}$$

Now, we have

$$X_c\, g_{ab} = g(\nabla_{X_c} X_a, X_b) + g(X_a, \nabla_{X_c} X_b),$$

and we get $X_c\, g_{ab}\big|_p = 0$. This yields the first order Taylor expansion

$$g_{ab} = \delta_{ab} + \mathcal{O}(|x|^2),$$

To compute the second order terms, it suffices to compute $X_c^2\, g_{ab}$ at p and polarize. We compute

$$X_c^2\, g_{ab} = g(\nabla^2_{X_c} X_a, X_b) + g(X_a, \nabla^2_{X_c} X_b) + 2\, g(\nabla_{X_c} X_a, \nabla_{X_c} X_b)$$

Using (6.20) we get

$$X_c^2\, g_{ab}\big|_p = \frac{2}{3}\, g(R(E_c, E_a)\, E_c, E_b).$$

The formula for the second order Taylor coefficient for g_{ab} now follows at once.

Similarly, we compute

$$\begin{aligned} X_c^3\, g_{ab}\big|_p &= g(\nabla^3_{X_c} X_a, X_b) + 3g(\nabla^2_{X_c} X_a, \nabla_{X_c} X_b) \\ &+ 3g(\nabla_{X_c} X_a, \nabla^2_{X_c} X_b) + g(X_a, \nabla^3_{X_c} X_b) \end{aligned}$$

and using (6.21) this gives

$$X_c^3\, g_{ab}\big|_p = g(\nabla_{E_c} R(E_a, E_c)\, E_b, E_c).$$

the formula for the second order Taylor expansion for g_{ab} holds at once. $\square$

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